Voorbeeldoplossingen:

1. Bereken de volgende limieten:

a) $\lim\limits_{x\to 0}\dfrac{e^x-1-x-\frac{x^2}{2}}{x^3}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{e^x-1-x}{3x^2}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{e^x-1}{6x}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{e^x}{6}=\dfrac{1}{6}$

b) $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2\sin x}{x-\sin x}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{e^x+e^{-x}-2\cos x}{1-\cos x}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}+2\sin x}{\sin x}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{e^x+e^{-x}+2\cos x}{\cos x}=4$

c) $\lim\limits_{x\to 0}\dfrac{x^3-6x+6\sin x}{x^5}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{3x^2-6+6\cos x}{5x^4}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{6x-6\sin x}{20x^3}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{6-6\cos x}{60x^2}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{6\sin x}{120x}\overset{*}{=}\dfrac{1}{20}$

$*$: $\lim\limits_{x\to 0}\dfrac{6\sin x}{120x}=\dfrac{1}{20}\cdot\underbrace{\lim\limits_{x\to 0}\dfrac{\sin x}{x}}_{=1}=\dfrac{1}{20}$

d) $\lim\limits_{x\to 0}\dfrac{\ln\left(\cos(2x)\right)}{x^2}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{\frac{-2\sin 2x}{\cos 2x}}{2x}\overset{*}{=}-2$

$*$: $\lim\limits_{x\to 0}\dfrac{\frac{-2\sin 2x}{\cos 2x}}{2x}=-2\underbrace{\lim\limits_{x\to 0}\dfrac{1}{\cos 2x}}_{=1}\cdot\underbrace{\lim\limits_{2x\to 0}\dfrac{\sin 2x}{2x}}_{=1}=-2$ (als $x\to 0$ dan zal ook $2x\to 0$)

e) $\lim\limits_{x\to 0}\dfrac{\text{Bgsin}\,x-x}{x^3}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{\frac{1}{\sqrt{1-x^2}}-1}{3x^2}=\lim\limits_{x\to 0}\dfrac{\left(1-x^2\right)^{-1/2}-1}{3x^2}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{-\frac{1}{2}\cdot\left(1-x^2\right)^{-3/2}\cdot\left(-2x\right)}{6x}=\dfrac{1}{6}$

f) $\lim\limits_{x\to 0}\dfrac{e^{x^2}-1-x^2}{x^4}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{2x\cdot e^{x^2}-2x}{4x^3}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{4x\cdot e^{x^2}}{8x}=\dfrac{1}{2}$

(je kan hier ook substitutie $t=x^2$ gebruiken: $\lim\limits_{t\to 0}\dfrac{e^t-1-t}{t^2}\overset{H}{=}\lim\limits_{t\to 0}\dfrac{e^t-1}{2t}\overset{H}{=}\lim\limits_{t\to 0}\dfrac{e^t}{2}=\dfrac{1}{2}$)

g) $\lim\limits_{x\to 1}\dfrac{\ln x-x+1}{(x-1)^2}\overset{H}{=}\lim\limits_{x\to 1}\dfrac{\frac{1}{x}-1}{2(x-1)}\overset{H}{=}\lim\limits_{x\to 1}\dfrac{\frac{-1}{x^2}}{2}=\dfrac{-1}{2}$

h) $\lim\limits_{x\to 0}\dfrac{x-\sin x}{\sin^3 x}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{1-\cos x}{3\cdot\sin^2 x\cdot\cos x}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{\sin x}{6\cdot\sin x\cdot\cos^2 x-3\sin^2 x}=\dfrac{1}{6}$

i) $\lim\limits_{x\to 0}\dfrac{x-\ln(1+x)}{x^2}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{1-\frac{1}{1+x}}{2x}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{-\frac{-1}{(1+x)^2}}{2}=\dfrac{1}{2}$

j) $\lim\limits_{x\to 0}\dfrac{\tan x-x}{x-\sin x}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{\frac{1}{\cos^2 x}-1}{1-\cos x}=\lim\limits_{x\to 0}\dfrac{1-\cos^2 x}{\cos^2 x(1-\cos x)}=\lim\limits_{x\to 0}\dfrac{(1-\cos x)(1+\cos x)}{\cos^2 x(1-\cos x)}=2$

k) $\lim\limits_{x\to 0}\dfrac{\text{Bgsin}(2x)-2\,\text{Bgsin}\,x}{x^3}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{\frac{2}{\sqrt{1-4x^2}}-\frac{2}{\sqrt{1-x^2}}}{3x^2}=\lim\limits_{x\to 0}\dfrac{2\left(1-4x^2\right)^{-1/2}-2\left(1-x^2\right)^{-1/2}}{3x^2}$

$$\overset{H}{=}\lim_{x\to 0}\frac{-\left(1-4x^2\right)^{-3/2}\cdot\left(-8\cancel{x}\right)+\left(1-x^2\right)^{-3/2}\cdot\left(-2\cancel{x}\right)}{6\cancel{x}}=1$$

l) $\lim\limits_{x\to 0}\dfrac{\pi-2\,\mathrm{Bgcos}\,x-2x}{x^3}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{\dfrac{2}{\sqrt{1-x^2}}-2}{3x^2}=\lim\limits_{x\to 0}\dfrac{-\left(1-x^2\right)^{-3/2}\cdot\left(-2\cancel{x}\right)}{6\cancel{x}}=\dfrac{1}{3}$

m) $\lim\limits_{x\to 0}\dfrac{\mathrm{Bgtan}\,x-\sin x}{x^3}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{\dfrac{1}{1+x^2}-\cos x}{3x^2}\overset{H}{=}\lim\limits_{x\to 0}\dfrac{\dfrac{-2x}{\left(1+x^2\right)^2}+\sin x}{6x}$

$$=\lim_{x\to 0}\left(-\frac{1}{3}\cdot\frac{1}{\left(1+x^2\right)^2}+\frac{1}{6}\cdot\frac{\sin x}{x}\right)=\frac{-1}{6}$$

n) $\lim\limits_{x\to 0}\dfrac{\mathrm{Bgtan}(x^2)-x^2}{x^6}\overset{t=x^2}{=}\lim\limits_{t\to 0}\dfrac{\mathrm{Bgtan}\,t-t}{t^3}\overset{H}{=}\lim\limits_{t\to 0}\dfrac{\dfrac{1}{1+t^2}-1}{3t^2}=\lim\limits_{t\to 0}\dfrac{-\cancel{t^2}}{\left(1+t^2\right)3\cancel{t^2}}=\dfrac{-1}{3}$

2. Bereken volgende limieten

a) $\lim\limits_{x\to+\infty}\left(\dfrac{2}{\pi}\mathrm{Bgtan}\,x\right)^x=L\overset{*}{=}e^{\ln L}=e^{-\frac{2}{\pi}}$

$$*:\ln L=\lim_{x\to+\infty}\left(x\cdot\ln\left(\frac{2}{\pi}\mathrm{Bgtan}\,x\right)\right)=\lim_{x\to+\infty}\frac{\ln\left(\frac{2}{\pi}\mathrm{Bgtan}\,x\right)}{1/x}\overset{H}{=}\lim_{x\to+\infty}\frac{\frac{1}{1+x^2}}{\frac{\mathrm{Bgtan}\,x}{-1/x^2}}=\lim_{x\to+\infty}\frac{-x^2}{\left(1+x^2\right)\cdot\mathrm{Bgtan}\,x}$$

$$=\lim_{x\to+\infty}\frac{-x^2}{1+x^2}\cdot\lim_{x\to+\infty}\frac{1}{\mathrm{Bgtan}\,x}=-\frac{2}{\pi}$$

b) $L=\lim\limits_{x\to+\infty}x^2\left(x\cdot\mathrm{Bgsin}\left(\dfrac{1}{x}\right)-1\right)$ stel $t=\dfrac{1}{x}$, dan zal $t\underset{>}{\to}0$ als $x\to+\infty$, dan wordt de limiet:

$$L=\lim_{t\underset{>}{\to}0}\frac{1}{t^2}\left(\frac{1}{t}\cdot\mathrm{Bg}\sin t-1\right)=\lim_{t\underset{>}{\to}0}\frac{\mathrm{Bg}\sin t-t}{t^3}\overset{\text{oef 1e}}{=}\frac{1}{6}$$

c) $\lim\limits_{x\to 0^+}(\mathrm{Bgsin}\,x)^{\tan x}=L\overset{*}{=}e^{\ln L}=e^0=1$

$$*:\ln L=\lim_{x\to 0^+}\tan x\cdot\ln\left(\mathrm{Bgsin}\,x\right)=\lim_{x\to 0^+}\frac{\ln\left(\mathrm{Bgsin}\,x\right)}{\cot x}\overset{H}{=}\lim_{x\to 0^+}\frac{\frac{1}{\sqrt{1-x^2}}}{\frac{\mathrm{Bgsin}\,x}{-1}}$$

$$=\underbrace{\lim_{x\to 0^+}\frac{-1}{\sqrt{1-x^2}}}_{=-1}\cdot\lim_{x\to 0^+}\frac{\sin^2 x}{\mathrm{Bgsin}\,x}\overset{H}{=}-\lim_{x\to 0^+}\frac{2\sin x\cos x}{\frac{1}{\sqrt{1-x^2}}}=0$$

d) $\lim_{x\to+\infty}\left(x^3\,\mathrm{Bgtan}\,x-\frac{\pi x^3}{2}+x^2\right)=\lim_{x\to+\infty}\frac{\mathrm{Bgtan}\,x-\frac{\pi}{2}+\frac{1}{x}}{\frac{1}{x^3}}\overset{H}{=}\lim_{x\to+\infty}\frac{\frac{1}{1+x^2}-\frac{1}{x^2}}{\frac{-3}{x^4}}$

$=\lim_{x\to+\infty}\frac{-x^4}{-3x^2\left(1+x^2\right)}=\frac{1}{3}$

e) $\lim_{x\to+\infty}\left(2-e^{1/x}\right)^x=L\overset{*}{=}e^{\ln L}=e^{-1}=\frac{1}{e}$

$*:\ln L=\lim_{x\to+\infty}x\cdot\ln\left(2-e^{1/x}\right)=\lim_{x\to+\infty}\frac{\ln\left(2-e^{1/x}\right)}{1/x}\overset{t=\frac{1}{x}}{=}\lim_{\substack{t\to0\\>}}\frac{\ln\left(2-e^{t}\right)}{t}\overset{H}{=}\lim_{\substack{t\to0\\>}}\frac{-e^t}{2-e^t}=-1$

f) $\lim_{x\to+\infty}\left(x^3+x^2+1\right)^{\frac{1}{\ln x}}=L\overset{*}{=}e^{\ln L}=e^3$

$*:\ln L=\lim_{x\to+\infty}\frac{\ln\left(x^3+x^2+1\right)}{\ln x}\overset{H}{=}\lim_{x\to+\infty}\frac{\frac{3x^2+2x}{x^3+x^2+1}}{\frac{1}{x}}=\lim_{x\to+\infty}\frac{3x^3+2x^2}{x^3+x^2+1}=3$

g) $\lim_{x\to0}\left(\frac{1}{x^2}-\frac{1}{\tan^2 x}\right)=\lim_{x\to0}\frac{\tan^2x-x^2}{x^2\cdot\tan^2x}\cdot\underbrace{\frac{\cos^2x}{\cos^2x}}_{=1}=\lim_{x\to0}\frac{\sin^2x-x^2\cos^2x}{x^2\cdot\sin^2x}\cdot\underbrace{\lim_{x\to0}\frac{\sin^2x}{x^2}}_{=1}$

$=\lim_{x\to0}\frac{\sin^2x-x^2\cos^2x}{x^4}=\lim_{x\to0}\frac{\left(\sin x-x\cos x\right)\cdot\left(\sin x+x\cos x\right)}{x^4}$

$=\lim_{x\to0}\frac{\sin x-x\cos x}{x^3}\cdot\lim_{x\to0}\frac{\sin x+x\cos x}{x}\overset{*}{=}\frac{1}{3}\cdot2=\frac{2}{3}$

$*:\lim_{x\to0}\frac{\sin x-x\cos x}{x^3}\overset{H}{=}\lim_{x\to0}\frac{\cancel{\cos x}-\cancel{\cos x}+\cancel{x}\sin x}{3\cancel{x^2}}=\frac{1}{3}\lim_{x\to0}\frac{\sin x}{x}=\frac{1}{3}$

$\lim_{x\to0}\frac{\sin x+x\cos x}{x}\overset{H}{=}\lim_{x\to0}\left(\frac{\sin x}{x}+\cos x\right)=1+1=2$

3. Verklaar de fout in de volgende berekening en bereken de limiet op de juiste manier:

$\lim_{x\to0}\frac{\cos x}{x+1}\overset{H}{=}\lim_{x\to0}\frac{\sin x}{1}=0.$

De regel van l'Hôpital mag niet gebruikt worden, want $\lim_{x\to0}\cos x=1$ en $\lim_{x\to0}\left(x+1\right)=1$, dus

$\lim_{x\to0}\frac{\cos x}{x+1}=1$

4. Iemand gebruikt correct de methode van l'Hôpital maar komt in een oneindige lus terecht:

$$L=\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}\overset{H}{=}\lim_{x\to-\infty}\frac{1}{\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}=\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x}\overset{H}{=}\lim_{x\to-\infty}\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}=L$$

Bereken deze limiet zonder de regel van l'Hôpital te gebruiken.

$$L=\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}=\lim_{x\to-\infty}\frac{x}{\sqrt{x^2\cdot\left(1+\frac{1}{x^2}\right)}}=\lim_{x\to-\infty}\frac{x}{|x|\cdot\sqrt{1+\frac{1}{x^2}}}=\lim_{x\to-\infty}\frac{\cancel{x}}{-\cancel{x}\cdot\sqrt{1+\frac{1}{x^2}}}=-1$$

5. Iemand doet de volgende berekening: $\lim_{x\to+\infty}\frac{x+\sin x}{x}\overset{H}{=}\lim_{x\to+\infty}\frac{1+\cos x}{1}$ en besluit dat beide limieten niet bestaan omdat $f(x)=1+\cos x$ blijft schommelen tussen 0 en 2.

   Bereken de limiet zonder l'Hôpital te gebruiken en verklaar waar de fout zit in de berekening.

   De regel van l'Hôpital mag enkel gebruikt worden als de limiet die je uitkomt bestaat, wat hier niet het geval is.

   Juist is: $\lim_{x\to+\infty}\frac{x+\sin x}{x}=\lim_{x\to+\infty}\left(\frac{x}{x}+\frac{\sin x}{x}\right)=1+1=2$

6. Iemand past herhaaldelijk de regel van l'Hôpital toe, maar de limieten worden alleen maar moeilijker:

$$\lim_{\substack{x\to0\\>}}\frac{e^{-\frac{1}{x}}}{x}\overset{H}{=}\lim_{\substack{x\to0\\>}}\frac{e^{-\frac{1}{x}}\cdot\frac{1}{x^2}}{1}=\lim_{\substack{x\to0\\>}}\frac{e^{-\frac{1}{x}}}{x^2}\overset{H}{=}\lim_{\substack{x\to0\\>}}\frac{e^{-\frac{1}{x}}\cdot\frac{1}{x^2}}{2x}=\frac{1}{2}\lim_{\substack{x\to0\\>}}\frac{e^{-\frac{1}{x}}}{x^3}\overset{H}{=}\frac{1}{2}\lim_{\substack{x\to0\\>}}\frac{e^{-\frac{1}{x}}\cdot\frac{1}{x^2}}{3x^2}=\frac{1}{6}\lim_{\substack{x\to0\\>}}\frac{e^{-\frac{1}{x}}}{x^4}=\ldots$$

   Bereken deze limiet op een andere manier.

   Gebruik de substitutie $t=\frac{1}{x}$, dan is $\lim_{\substack{x\to0\\>}}\frac{e^{-\frac{1}{x}}}{x}=\lim_{t\to+\infty}\frac{e^{-t}}{1/t}=\lim_{t\to+\infty}\frac{t}{e^t}\overset{H}{=}\lim_{t\to+\infty}\frac{1}{e^t}=0$.